Generalized Frobenius Number of Three Variables

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Abstract

Have you ever thought about the largest number of Chicken McNuggets that can't be bought with available pack sizes? This simple question takes us into the study of Frobenius numbers: Let $A = (a_1, a_2, \ldots, a_k)$ be a k-tuple of positive integers and $s \geq 0$. The generalized Frobenius number $g(A; s)$ is the largest integer that has at most s representations in terms of $a_1, a_2, \ldots, a_k$ with non-negative integer coefficients. In this talk, we present a formula for the generalized Frobenius number in the case $k = 3$ under certain conditions.

1 Introduction

The purpose of this note is to provide an overview of the results obtained by the author in [14] and [15] on generalized Frobenius numbers in three variables. For integers $n \geq 1$, $k \geq 2$, let $A = (a_1, a_2, \ldots, a_k)$ be a k-tuple of positive integers and let $d(n; A) = d(n; a_1, a_2, \ldots, a_k)$ be the number of representations to $a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$. Its generating series is given by

$$\sum_{n\geq 0} d(n; a_1, \ldots, a_k)x^n = \frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_k})}.$$

Sylvester [16] and Cayley [6] show that $d(n; a_1, a_2, \ldots, a_k)$ can be expressed as the sum of a polynomial in n of degree $k - 1$ and a periodic function of period $a_1a_2 \cdots a_k$. Using Bernoulli numbers, Beck, Gessel, and Komatsu [1] derive the explicit formula for the polynomial section. Tripathi [19] provides a formula for $d(n; a_1, a_2)$. Komatsu [8] shows that the periodic function part is defined in terms of trigonometric functions for three variables in the pairwise coprime case.

There is the well-known linear Diophantine problem, posed by Sylvester [18], known as the *Frobenius problem*[1]: Given positive integers $a_1, a_2, \ldots, a_k$ such that $\gcd(a_1, a_2, \ldots, a_k) = 1$, find the largest integer that *cannot* be expressed as a non-negative integer linear combination of these numbers. *The largest integer* is called the *Frobenius number* of the tuple $A = (a_1, a_2, \ldots, a_k)$, and is denoted by $g(A) = g(a_1, a_2, \ldots, a_k)$. The Frobenius number of $A = (a_1, a_2, \ldots, a_k)$ exists if and only if $a_1, a_2, \ldots, a_k$ are relatively prime i.e. $\gcd(a_1, a_2, \ldots, a_k) = 1$ see for example [13]. With the above notation, the Frobenius number is given by

$$g(A) = \max\{n \in \mathbb{Z} \mid d(n; A) = 0\}.$$

For instance, at McDonald's one can only order packs of 6, 9, or 20 Chicken McNuggets. The following list shows the numbers of nuggets that cannot be purchased by ordering any amounts

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[1] It is also known as the coin problem, postage stamp problem, or Chicken McNugget problem, involves determining the largest value that cannot be formed using only coins of specified denominations.

of these packs (integers that cannot be expressed by 6, 9, and 20): 1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 37, and 43. Therefore, $g(6, 9, 20) = 43$.

Note that if all non-negative integers can be expressed as a non-negative integer linear combination of A, then $g(A) = -1$. For example, $g(1, 2) = -1$.

For two variables $A = \{a, b\} \subset \mathbb{Z}_{>0}$, it is shown by Sylvester [17] that

$$g(a, b) = ab - a - b. \tag{1}$$

For example, consider $A = (a, b) = (3, 5)$. Then the Frobenius number of A is given by $g(3, 5) = 15 - 3 - 5 = 7$, which means that all integers $n > 7$ can be expressed as a non-negative integer linear combination of 3 and 5.

Tripathi [20] has provided explicit but complicate formulas for calculating the Frobenius number in three variables. However, it is important to note that closed-form solutions for the general case become increasingly challenging when the number of variables exceeds three ($k > 3$). Nevertheless, various formulas have been proposed for Frobenius numbers in specific scenarios or special cases. For example, explicit formulas in some particular cases of sequences, including arithmetic, geometric-like, Fibonacci, Mersenne, and triangular (see [12] and references therein) are known.

In this note, we will focus on a generalization of the Frobenius number. For a given non-negative integer s, let

$$g(A; s) = g(a_1, a_2, \dots, a_k; s) = \max\{n \in \mathbb{Z} \mid d(n; A) \le s\}$$

be the largest integer such that the number of expressions that can be represented by $a_1, a_2, \dots, a_k$ is at most s. Notice that $g(a_1, a_2, \dots, a_k) = g(a_1, a_2, \dots, a_k; 0)$. That means all integers bigger than $g(A; s)$ have at least $s + 1$ representations. The $g(A; s)$ is called *the generalized Frobenius number*. Furthermore, $g(A; s)$ is well-defined (i.e. bounded above) (see [7]).

As a generalization of (1), for $A = (a, b)$ and $s \in \mathbb{Z}_{\ge 0}$, (see [3]), an exact formula for $g(A, s) = g(a, b; s)$ is given by

$$g(a, b; s) = (s + 1)ab - a - b. \tag{2}$$

In general, we have $d\big(g(A; s); A\big) \le s$, but in the case $|A| = 2$ one can show that actually $d\big(g(A; s); A\big) = s$. Similar to the $s = 0$ case, exact formulas for the generalized Frobenius number in the cases $k \ge 3$ are still unknown. For $k = 3$ exact formulas are just known for special cases. For example, there are explicit results in the case of triangular numbers [10], repunits [9] and Fibonacci numbers [11]. Recently, Binner [5] provide bounds for the number of solutions $a_1x_1 + a_2x_2 + a_3x_3 = n$ and use these bounds to solve $g(a_1, a_2, a_3; s)$ when s is large. In 2022, Woods [21] provide formulas and asymptotics for the generalized Frobenius problem using the restricted partition function.

One of main results is the following explicit formula for a special case of the generalized Frobenius number in three variables [14].

Theorem 1 ([14]). *Let a_1, a_2 and a_3 be positive integers with $\gcd(a_1, a_2, a_3) = 1$ and let $s \in \mathbb{Z}_{\ge 0}$. If $d_1 = gcd(a_2, a_3)$ and suppose that $a_1 \equiv 0 \pmod{\frac{a_2}{d_1}}$ or $a_1 \equiv 0 \pmod{\frac{a_3}{d_1}}$, then*

$$g\left(a_1, a_2, a_3; \sum_{j=0}^{s} \left\lceil \frac{j a_2 a_3}{a_1 d_1^2} \right\rceil \right) = (s + 1)\frac{a_2 a_3}{d_1} + a_1 d_1 - a_1 - a_2 - a_3.$$

Remark 2. In Theorem 1, the order of the integers in the tuple (a_1, a_2, a_3) does not matter due to the symmetry of g. So, if $d_2 = \gcd(a_1, a_3)$ and $a_2 \equiv 0 \pmod{\frac{a_1}{d_2}}$ or $a_2 \equiv 0 \pmod{\frac{a_3}{d_2}}$, then

$$g\left(a_1, a_2, a_3; \sum_{j=0}^{s} \left\lceil \frac{ja_1a_3}{a_2d_2^2} \right\rceil\right) = (s+1)\frac{a_1a_3}{d_2} + a_2d_2 - a_1 - a_2 - a_3.$$

Similarly, if $d_3 = \gcd(a_1, a_2)$ and $a_3 \equiv 0 \pmod{\frac{a_1}{d_3}}$ or $a_3 \equiv 0 \pmod{\frac{a_2}{d_3}}$, then

$$g\left(a_1, a_2, a_3; \sum_{j=0}^{s} \left\lceil \frac{ja_1a_2}{a_3d_1^2} \right\rceil\right) = (s+1)\frac{a_1a_2}{d_3} + a_3d_3 - a_1 - a_2 - a_3.$$

Remark 3. Notice that

$$\mathbf{U}_{(a_1,a_2,a_3)} := \bigcup_{i=1}^{3} \left\{ \sum_{j=0}^{s} \left\lceil \frac{j\prod_{\substack{1\le \ell \le 3\\ \ell \ne i}} a_\ell}{a_i d_i^2} \right\rceil \mid s \ge 0 \right\} \subseteq \{d(n; a_1, a_2, a_3) \mid n \in \mathbb{Z}_{>0}\}$$

and in general the left set is a proper subset of the right. For example, $d(120; 10, 15, 21) = 6$ but

$$6 \notin \mathbf{U}_{(10,15,21)} = \{0, 1, 2, 3, 4, 5, 7, 9, 11, 14, 17, 20, 22, 24, \dots\} \subsetneq \{d(n; 10, 15, 21) \mid n \in \mathbb{Z}_{>0}\}.$$

However, they are equal in some cases. For example, if a_1, a_2 and a_3 are of the form in [2], then we obtain that

$$\mathbf{U}_{(a_1,a_2,a_3)} = \{t_k \mid k \in \mathbb{Z}_{\ge 0}\} = \{d(n; a_1, a_2, a_3) \mid n \in \mathbb{Z}_{>0}\},$$

where t_k is the kth triangular number which is given by $t_k = \binom{k+1}{2}$.

We will give the proof of Theorem 1 in Section 3.

The other main result is to show the explicit formula for three consecutive triangular numbers which is $g(t_n, t_{n+1}, t_{n+2}; s)$ for all $s \ge 0$ and for all sufficiently large n (depends on s).

Theorem 4 ([15]). *The $g(t_n, t_{n+1}, t_{n+2}; s)$ are given for all $s \ge 0$ as follows:*

(i) For even $n > 6\lfloor\sqrt{s+1}\rfloor - 6$, we have

$$g(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n+1)(n+2)}{4}(q_s n + 6c_s) - 1.$$

(ii) For odd $n > 6\left\lfloor \frac{\sqrt{4s+5}-1}{2} \right\rfloor - 3$, we have

$$g(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n+1)(n+2)}{4}(q_s n + 6c_s - 3\delta_s) - 1.$$

Here the q_s, c_s and δ_s are given by

$$q_s = 2\lfloor\sqrt{s}\rfloor + 2 + \delta_s, \quad c_s = s - \lfloor\sqrt{s}\rfloor^2 - \delta_s\lfloor\sqrt{s}\rfloor, \qquad \delta_s = \begin{cases} 1 & \text{if } s \ge \lfloor\sqrt{s}\rfloor^2 + \lfloor\sqrt{s}\rfloor, \\ 0 & \text{else.} \end{cases}$$

We define $\mathbb{B} = \{n \in \mathbb{Z}_{\ge 1} \mid n = k^2 \text{ or } n = k(k+1), \exists k \ge 1\} = \{1, 2, 4, 6, 9, 12, 16, \dots\}$. Then, we obtain the following corollary whose main idea is to derive a formula for $g(t_n, t_{n+1}, t_{n+2}; s+1)$ based on $g(t_n, t_{n+1}, t_{n+2}; s)$ when $s \ge 0$.

Corollary 5. *Let $s \in \mathbb{Z}_{\geq 0}$. Then the following statements hold:*

(i) If $s+1 \notin \mathbb{B}$, we have

$$g(t_n, t_{n+1}, t_{n+2}; s+1) - g(t_n, t_{n+1}, t_{n+2}; s) = \frac{6(n+1)(n+2)}{4}.$$

(ii) If n is even and $s+1 \in \mathbb{B}$, that is $s+1 = k^2$ or $s+1 = k(k+1)$ $(\exists k \geq 1)$, then

$$g(t_n, t_{n+1}, t_{n+2}; s+1) - g(t_n, t_{n+1}, t_{n+2}; s) = \frac{(n-6k+6)(n+1)(n+2)}{4}.$$

(iii) If n is odd and $s+1 \in \mathbb{B}$, then

$$g(t_n, t_{n+1}, t_{n+2}; s+1) - g(t_n, t_{n+1}, t_{n+2}; s) = \begin{cases} \frac{(n-6k+9)(n+1)(n+2)}{4} & \text{if } s+1 = k^2, \\ \frac{(n-6k+3)(n+1)(n+2)}{4} & \text{if } s+1 = k(k+1). \end{cases}$$

2 Preliminary Lemmas

Before proving Theorem 1, we introduce some Lemmas. Beck and Kifer [2] show the following result on $g(a_1, a_2, \ldots, a_k; s)$ in terms of $\ell = \gcd(a_2, a_3, \ldots, a_k)$.

Lemma 6 ([2, Lemma 4])**.** *For $k \geq 2$, let $A = (a_1, \ldots, a_k)$ be a k-tuple of positive integers with* $\gcd(A) = 1$. *If $\ell = \gcd(a_2, a_3, \ldots, a_k)$, let $a_j = \ell a'_j$ for $2 \leq j \leq k$. Then for $s \geq 0$*

$$g(a_1, a_2, \ldots, a_k; s) = \ell\, g\big(a_1, a'_2, a'_3, \ldots, a'_k; s\big) + a_1(\ell - 1).$$

The next lemma give an upper bound for the number of representations to $a_1x_1 + \cdots + a_kx_k = g(a_1, \ldots, a_k; s) - jc$, for all integers j such that $0 \leq jc \leq g(a_1, \ldots, a_k; s)$ when $c \equiv 0 \pmod{a_r}$ for some $r \in \{1, \ldots, k\}$.

Lemma 7. *For $k \geq 2$, let $A = (a_1, \ldots, a_k)$ be a k-tuple of positive integers with* $\gcd(A) = 1$ *and let $s \in \mathbb{Z}_{\geq 0}$. If c is a positive integer such that $c \equiv 0 \pmod{a_r}$ for some $r = 1, \ldots, k$, then, for all integers $0 \leq jc \leq g(A; s)$,*

$$d\big(g(A; s) - jc; A\big) \leq s.$$

Proof. Suppose that $c \in \mathbb{Z}_{>0}$ such that $c \equiv 0 \pmod{a_r}$ for some $r = 1, \ldots, k$. Assume that there exists $0 \leq j \leq \frac{g(A;s)}{c}$ such that

$$d\big(g(A; s) - jc; A\big) \geq s + 1.$$

So, there are *at least* $s+1$ non-negative integer solutions $(x_1, \ldots, x_k)$ such that

$$g(A; s) - jc = \sum_{\ell=1}^{k} x_\ell a_\ell.$$

Since $c \equiv 0 \pmod{a_r}$, then $c = a_r q$ for some $q \in \mathbb{Z}_0$. So, we obtain that

$$g(A; s) = x_1 a_1 + \ldots + x_{r-1} a_{r-1} + (x_r + jq) a_r + x_{r+1} a_{r+1} + \ldots + x_k a_k,$$

this means that $g(A; s)$ has *at least* $s+1$ non-negative representations in terms of $a_1, \ldots, a_k$. We get a contradiction since $g(A; s)$ must have at most s representations. $\square$

To accomplish the proof of Theorem 1, we need the following lemma. If $k = 2$, says $A = \{a, b\}$, then, for a non-negative number $j \le g(a, b; s)/c$, $d\big(g(a, b; s) + jc; a, b\big) = i$ is equivalent to $g(a, b; i - 1) < g(a, b; s) + jc \le g(a, b; i)$.

Lemma 8. *Let $a, b \in \mathbb{Z}_{>0}$ with $\gcd(a, b) = 1$, and let $i, s \in \mathbb{Z}_{\ge 0}$. Suppose that c is a positive integer such that $c \equiv 0 \pmod a$ or $c \equiv 0 \pmod b$ and $j \in \mathbb{Z}$. Then*

$$d\big(g(a, b; s) + jc; a, b\big) = i,$$

if and only if,

$$g(a, b; i - 1) < g(a, b; s) + jc \le g(a, b; i).$$

Here we set $g(a, b; -1)$ to be -2.

Proof. Without loss of generality, we assume that $c \equiv 0 \pmod a$. For convenient, throughout the proof, for $s \ge 0$, we denote $g(s) := g(a, b; s)$, which is $g(s) = (s + 1)ab - a - b$.

($\Rightarrow$) Suppose that $d(g(s) + jc; a, b) = i$. By the definition of $g(i) = g(a, b; i)$, it follows immediately that $g(s) + jc \le g(a, b; i)$. Clearly, if $i = 0$, then $-2 < g(s) + jc \le g(0)$, we are done. So, assume that $i \ge 1$. Obviously, $g(s) + jc \ne g(i - 1)$, otherwise $d(g(s) + jc; a, b) = i - 1$, a contradiction. It remains to show that

$$g(i - 1) < g(s) + jc.$$

We will prove this statement by assuming that $g(s) + jc < g(i - 1)$. So if we set $\Delta = g(i - 1) - (g(s) + jc)$, then $\Delta > 0$, moreover Δ is a positive integer that is a multiple of a by using the hypothesis $c \equiv 0 \pmod a$. Thus

$$g(i - 1) = g(s) + jc + \Delta,$$

this implies that $g(i - 1) = g(a, b; i - 1)$ has at least i representations in terms of a and b, a contradiction with the definition of $g(a, b; i - 1)$.

($\Leftarrow$) Suppose that $g(i - 1) < g(s) + jc \le g(i)$ for some $i \in \mathbb{Z}_{\ge 0}$. Clearly, since $g(i - 1) < g(s) + jc$, we have

$$d(g(s) + jc; a, b) \ge i.$$

Our goal is to show that $d(g(s) + jc; a, b) = i$. If $g(s) + jc = g(i)$, then we are done. So we assume that $g(s) + jc < g(i)$ and also assume that $d(g(s) + jc; a, b) > i$. With the same setting, let $\Delta = g(i) - (g(s) + jc)$. Then $\Delta > 0$. In addition, Δ is a multiple of a. So we obtain that

$$g(i) = g(s) + jc + \Delta,$$

i.e., $g(i)$ has *at least* $i + 1$ representations in terms of a and b, a contradiction. Therefore, $d(g(s) + jc; a, b) = i$. Similarly to that, we can prove the case where $c \equiv 0 \pmod b$. □

Lemma 9. *Let $a, b \in \mathbb{Z}_{>0}$ with $a < b$, $\gcd(a, b) = 1$, and let $s, K \in \mathbb{Z}_{\ge 0}$. If m is an integer such that $m > g(a, b; s) + Ka$, then, for all $j \in \mathbb{Z}_{\ge 0}$, we have*

$$d\big(m - ja; a, b\big) \ge d\big(g(a, b; s) + (K - j)a; a, b\big).$$

Proof. We will prove by induction on j. If $j = 0$, we can assume that there exists a non-negative integer ℓ such that $d\big(g(a, b; s) + Ka; a, b\big) = \ell$. By Lemma 8, we have

$$g(a, b; \ell - 1) < g(a, b; s) + Ka \le g(a, b; \ell).$$

Hence $m > g(a, b; \ell - 1)$, which means

$$d(m; a, b) \geq \ell = d\big(g(a, b; s) + Ka; a, b\big).$$

The base step is proved.

Let j be a non-negative integer and assume that

$$d\big(m - ja; a, b\big) \geq d\big(g(a, b; s) + (K - j)a; a, b\big).$$

We want to show that

$$d\big(m - (j + 1)a; a, b\big) \geq d\big(g(a, b; s) + (K - (j + 1))a; a, b\big).$$

Assume that $d\big(m-(j+1)a; a, b\big) < d\big(g(a, b; s)+(K-(j+1))a; a, b\big)$. Suppose that there exists $\ell \geq 0$ such that $d\big(m - (j + 1)a; a, b\big) = \ell$. So

$$\ell < d\big(g(a, b; s) + (K - (j + 1))a; a, b\big).$$

Observe that $d\big(g(a, b; s) + (K - (j + 1))a; a, b\big) \leq d\big(g(a, b; s) + (K - j)a; a, b\big)$ by Lemma 8. One can see that since $d\big(m - (j + 1)a; a, b\big) = \ell$, then

$$d\big(m - ja; a, b\big) = \begin{cases} \ell + 1 & \text{if } b \mid (m - ja), \\ \ell & \text{otherwise.} \end{cases}$$

If $d\big(m - ja; a, b\big) = \ell$, then we get a contradiction that

$$\ell = d\big(m - ja; a, b\big) \geq d\big(g(a, b; s) + (K - j)a; a, b\big) \geq d\big(g(a, b; s) + (K - (j + 1))a; a, b\big) > \ell.$$

If $d\big(m - ja; a, b\big) = \ell + 1$, then

$$\ell < d\big(g(a, b; s) + (K - (j + 1))a; a, b\big) \leq d\big(g(a, b; s) + (K - j)a; a, b\big) \leq d\big(m - ja; a, b\big) = \ell + 1.$$

It follows that

$$d\big(g(a, b; s) + (K - (j + 1))a; a, b\big) = d\big(g(a, b; s) + (K - j)a; a, b\big) = d\big(m - ja; a, b\big) = \ell + 1.$$

Since $d\big(g(a, b; s)+(K-(j+1))a; a, b\big) = \ell+1$, by Lemma 8, $g(a, b; s)+(K-(j+1))a > g(a, b; \ell)$. This means $m - (j + 1)a > g(a, b; s) + (K - (j + 1))a > g(a, b; \ell)$, thus

$$d\big(m - (j + 1)a; a, b\big) \geq \ell + 1,$$

which contradicts with $d(m - (j + 1)a; a, b) = \ell$. Therefore,

$$d\big(m - (j + 1)a; a, b\big) \geq d\big(g(a, b; s) + (K - (j + 1))a; a, b\big),$$

as required. □

3 Proof of Theorem 1

By applying Lemma 6, Lemma 7 and Lemma 8, we can prove Theorem 1 as follows.

Proof of Theorem 1. Suppose that $d_1 = \gcd(a_2, a_3)$ and $a_1 \equiv 0 \pmod{\frac{a_2}{d_1}}$. By applying Lemma 6, we obtain that

$$g\left(a_1, a_2, a_3; \sum_{j=0}^{s} \left\lceil \frac{ja_2a_3}{a_1d_1^2} \right\rceil\right) = d_1 g\left(a_1, \frac{a_2}{d_1}, \frac{a_3}{d_1}; \sum_{j=0}^{s} \left\lceil \frac{ja_2a_3}{a_1d_1^2} \right\rceil\right) + a_1(d_1 - 1). \tag{3}$$

We will show that

$$g\left(a_1, \frac{a_2}{d_1}, \frac{a_3}{d_1}; \sum_{j=0}^{s} \left\lceil \frac{ja_2a_3}{a_1d_1^2} \right\rceil\right) = g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right).$$

Then one can see that, for $m \in \mathbb{Z}_{\geq 0}$,

$$d\left(m; a_1, \frac{a_2}{d_1}, \frac{a_3}{d_1}\right) = \sum_{j=0}^{\lfloor \frac{m}{a_1} \rfloor} d\left(m - ja_1; \frac{a_2}{d_1}, \frac{a_3}{d_1}\right). \tag{4}$$

Put $m = g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right)$, then we obtain that

$$d\left(g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right); a_1, \frac{a_2}{d_1}, \frac{a_3}{d_1}\right) = \sum_{j=0}^{\left\lfloor \frac{g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right)}{a_1} \right\rfloor} d\left(g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right) - ja_1; \frac{a_2}{d_1}, \frac{a_3}{d_1}\right). \tag{5}$$

By Lemma 7, we have that each value of $d\left(g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right) - ja_1; \frac{a_2}{d_1}, \frac{a_3}{d_1}\right)$ have to be equal to any of $0, 1, \dots, s$. To calculate the right-hand side of (5), we count the number of $0 \leq j \leq g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right)/a_1$ such that

$$d\left(g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right) - ja_1; \frac{a_2}{d_1}, \frac{a_3}{d_1}\right) = i, \tag{6}$$

for all $i = 1, 2, \dots, s$. For convenient, let $g_s := g\left(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\right)$. For given i the j such that (6) holds are, by Lemma 8, those with $g_{i-1} < g_s - ja_1 \leq g_i$. By (2), this is equivalent to

$$\frac{ia_2a_3}{d_1^2} - \frac{a_2}{d_1} - \frac{a_3}{d_1} < (s+1)\frac{a_2a_3}{d_1^2} - \frac{a_2}{d_1} - \frac{a_3}{d_1} - ja_1 \leq (i+1)\frac{a_2a_3}{d_1^2} - \frac{a_2}{d_1} - \frac{a_3}{d_1}.$$

So,

$$(s+1-i)\frac{a_2a_3}{a_1d_1^2} > j \geq (s-i)\frac{a_2a_3}{a_1d_1^2}.$$

Thus, by Lemma 8, there are

$$\left\lceil (s+1-i)\frac{a_2a_3}{a_1d_1^2} \right\rceil - \left\lceil (s-i)\frac{a_2a_3}{a_1d_1^2} \right\rceil$$

of j in $[0, g_s/a_1)$ such that $d\big(g_s - ja_1; \frac{a_2}{d_1}, \frac{a_3}{d_1}\big) = i$ for $i = 1, 2, \dots, s$. So, by (5), we have

$$\begin{aligned}
& d\left(g\Big(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\Big); a_1, \frac{a_2}{d_1}, \frac{a_3}{d_1}\right) \\
&= \sum_{j=0}^{\left\lfloor \frac{g_s}{a_1} \right\rfloor} d\left(g\Big(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\Big) - ja_1; \frac{a_2}{d_1}, \frac{a_3}{d_1}\right) \\
&= s\left\lceil \frac{a_2 a_3}{a_1 d_1^2} \right\rceil + (s-1)\left(\left\lceil \frac{2a_2 a_3}{a_1 d_1^2} \right\rceil - \left\lceil \frac{a_2 a_3}{a_1 d_1^2} \right\rceil\right) + (s-2)\left(\left\lceil \frac{3a_2 a_3}{a_1 d_1^2} \right\rceil - \left\lceil \frac{2a_2 a_3}{a_1 d_1^2} \right\rceil\right) + \\
&\quad \cdots + \left(\left\lceil \frac{s a_2 a_3}{a_1 d_1^2} \right\rceil - \left\lceil \frac{(s-1) a_2 a_3}{a_1 d_1^2} \right\rceil\right) \\
&= \sum_{j=0}^{s} \left\lceil \frac{j a_2 a_3}{a_1 d_1^2} \right\rceil .
\end{aligned}$$

Therefore, by the choice of $m = g\Big(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\Big)$ and (4) combined with Lemma 9, this value m is the largest that the right-hand side of (4) is (less than or) equal to $\sum_{j=0}^{s} \left\lceil \frac{j a_2 a_3}{a_1 d_1^2} \right\rceil$. Then

$$g\Big(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\Big) = g\left(a_1, \frac{a_2}{d_1}, \frac{a_3}{d_1}; \sum_{j=0}^{s} \left\lceil \frac{j a_2 a_3}{a_1 d_1^2} \right\rceil\right).$$

Hence, by (3) and (2),

$$\begin{aligned}
g\left(a_1, a_2, a_3; \sum_{j=0}^{s} \left\lceil \frac{j a_2 a_3}{a_1 d_1^2} \right\rceil\right) &= d_1 g\Big(\frac{a_2}{d_1}, \frac{a_3}{d_1}; s\Big) + a_1(d_1 - 1) \\
&= d_1\Big((s+1)\frac{a_2 a_3}{d_1^2} - \frac{a_2}{d_1} - \frac{a_3}{d_1}\Big) + a_1 d_1 - a_1 \\
&= (s+1)\frac{a_2 a_3}{d_1} + a_1 d_1 - a_1 - a_2 - a_3.
\end{aligned}$$

□

Compared to the results in [9, 11] our main theorem seems more useful when s is large, since their results have an upper bound on s. The result in [4] holds for s is extremely large. For example, by [4, Section 3.2], $g(16, 23, 37; s)$ can be found for $s \ge 157291918$. Therefore, our result behaves nicely for s not too large. In [21] the value for s is not explicitly given.

4 Proof of Theorem 4

In this section, we give a sketch of the proof of Theorem 4. For this, we define x_s^{even}, y_s^{even}, x_s^{odd}, and y_s^{odd} as follows.

Definition 10. Let s be a non-negative integer and let k be the non-negative integer such that

$$s = k(k+1) + i,$$

for some $i \in \{0,1,\ldots,2k+1\}$. Then we define integers x_s^{even}, y_s^{even}, x_s^{odd}, and y_s^{odd} as follows:

$$\left(x_s^{even}, y_s^{even}\right) = \begin{cases} \left(i, 2(k-i)\right) & \text{if } 0 \le i \le k, \\ \left(i-k-1, 4k-2i+3\right) & \text{if } k+1 \le i \le 2k+1, \end{cases}$$

$$\left(x_s^{odd}, y_s^{odd}\right) = \begin{cases} \left(2i, k-i\right) & \text{if } 0 \le i \le k, \\ \left(2(i-k)-1, 2k-i+1\right) & \text{if } k+1 \le i \le 2k+1, \end{cases}$$

Sketch of proof of Theorem 4. By using Lemma 6 we obtain that for $s \ge 0$

$$g\Big(t_n, t_{n+1}, t_{n+2}; s\Big) = d_1 g\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big) + t_n(d_1 - 1),$$

where $d_1 = \gcd(t_{n+1}, t_{n+2})$. One can show that, for all even $n > 6\lfloor\sqrt{s+1}\rfloor - 6$ and for all odd $n > 6\left\lfloor \frac{\sqrt{4s+5}-1}{2} \right\rfloor - 3$,

$$\begin{aligned} g\Big(t_n, \frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; s\Big) &= g\Big(\frac{t_{n+1}}{d_1}, \frac{t_{n+2}}{d_1}; x_s\Big) + y_s t_n \\ &= (x_s + 1)\frac{t_{n+2}t_{n+1}}{d_1^2} - \frac{t_{n+2}}{d_1} - \frac{t_{n+1}}{d_1} + y_s t_n, \end{aligned}$$

where $(x_s, y_s) = (x_s^{even}, y_s^{even})$ if n is even and $(x_s, y_s) = (x_s^{odd}, y_s^{odd})$ if n is odd. Using this, one can then show by direct calculation that the statement in Theorem 4 follows. □

References

[1] Beck, M., Gessel, I.M., Komatsu, T.: The polynomial part of a restricted partition function related to the Frobenius problem, Electron. J. Combin. **8**(No.1), Note 7, 1–5 (2001)

[2] Beck, M., Kifer, C.: An extreme family of generalized Frobenius numbers, Integers **11**, A24, 639—645 (2011)

[3] Beck, M., Robins, S.: A formula related to the Frobenius problem in two dimensions. Number theory (New York Seminar 2003), Springer, NewYork, 17–23 (2004)

[4] Binner, D.S.: The number of solutions to $ax + by + cz = n$ and its relation to quadratic residues, J. Integer Seq. **23**, Article 20.6.5. (2020)

[5] Binner, D.S.: Some Bounds for Number of Solutions to $ax + by + cz = n$ and their Applications, preprint arXiv:2106.13796, (2021)

[6] Cayley, A.: On a problem of double partitions, Philos. Mag. **XX**, 337—341 (1860)

[7] Fukshansky, L., Schurmann, A.: Bounds on generalized Frobenius numbers, Eur. J. Comb., **32**, 361—368 (2011)

[8] Komatsu, T.: On the number of solutions of the Diophantine equation of Frobenius–General case, Math. Commun. **8**, 195—206 (2003)

[9] Komatsu, T.: The Frobenius number associated with the number of representations for sequences of repunits, C. R. Acad. Sci. Paris **361**, 73–89 (2023)

[10] Komatsu, T.: The Frobenius number for sequences of triangular numbers associated with number of solutions, Ann. Comb. **26**, no. 3, 757–779 (2022)

[11] Komatsu, T., Ying, H.: The Frobenius number associated with the number of representations for Fibonacci triplets. Math. Biosci. Eng. **20**, 3455–3481 (2023)

[12] Robles-Pérez, A.M., Rosales, J.C.: The Frobenius number for sequences of triangular and tetrahedral numbers, J. Number Theory **186**, 473–492 (2018)

[13] Rosales, J.C., García-Sánchez, P.A.: Numerical semigroups, Developments in Mathematics, Vol., 20. Springer, New York (2009)

[14] Subwattanachai, K.: Generalized Frobenius Number of Three Variables, preprint arXiv:2309.09149 (2023)

[15] Subwattanachai, K.: Formulas for the Generalized Frobenius Number of Triangular Numbers, in preparation.

[16] Sylvester, J.J.: On the partition of numbers, Quart. J. Pure Appl. Math. **1**, 141–152 (1857)

[17] Sylvester, J.J.: On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order, Amer. J. Math. **5**, 119–136 (1882)

[18] Sylvester, J.J.: Mathematical questions with their solutions, Educational Times *41*, 21 (1884)

[19] Tripathi, A.: The number of solutions to $ax + by = n$, Fibonacci Quart. **38**, 290–293 (2000)

[20] Tripathi, A.: Formulae for the Frobenius number in three variables, J. Number Theory **170**, 368–389 (2017)

[21] Woods, K.: The generalized Frobenius problem via restricted partition functions. preprint arXiv:2011.00600 (2022)